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Extending the Conception of Horizon Content Knowledge: Fundamental Mathematical Knowledge and Its Sample Assessment Item

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Abstract

The purpose of this article is to extend the concept of Horizon Content Knowledge and provide a sample assessment item. Fundamental mathematical knowledge (FMK) is defined as an awareness of the core or in-depth mathematics behind solving a problem. Teachers with fundamental mathematical knowledge can provide students with a more broad and comprehensive approach to solving mathematical problems that extends beyond the rote procedures used to solve a particular problem. We propose that this type of knowledge should be included in the model of Horizon Content Knowledge. Developing items that assess teachers' Horizon Content Knowledge is worthwhile because this process provides a practical way to illuminate and refine the structure and content of Horizon Content Knowledge.

Key words: Assessment, Fundamental Mathematical Knowledge (FMK), Horizon Content Knowledge (HCK), Mathematics Knowledge for Teaching (MKT), Teacher Knowledge

Although recent research has established the concept of Horizon Content Knowledge (HCK) (Ball & Bass, 2009; Foster, 2011; Zazkis & Mamolo, 2011), issues regarding the definition of HCK, the incorporation of HCK into teaching practice and the assessment of HCK continue to be debated (e.g., Figueiras, Riberio, Chrrillo, Fernández, & Deulofeu, 2011; Jakobsen, Thames, & Riberio, 2013; Zazkis & Mamolo, 2012). Based on prior research and our research on Taiwanese high school teachers' HCK (Lin & Chin, 2014), this article attempts to expand the conceptualization of HCK and to complement earlier models by incorporating the dimension of "Fundamental Mathematical Knowledge" (FMK).

In contrast to Zazkis and Mamolo's horizon of "an advanced perspective on elementary knowledge" (2011, p. 13), our focus is a horizon that is related to "a kind of elementary perspective on advanced knowledge" (Ball & Bass, 2009, p. 10). In the present article, the concept of FMK primarily focuses on the students' horizon and how school mathematics may provide a basis for later (or advanced) mathematics knowledge rather than di-

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rectly applying advanced mathematical knowledge to school mathematics.

Horizon Content Knowledge

HCK, which was first proposed by Ball (1990) and later incorporated into the model of mathematics knowledge for teaching (Ball, Thames, & Phelps, 2008), is defined as the understanding of the broader set of mathematical ideas in which a particular mathematical concept is connected (Ball et al.). In contrast to Klein's idea of a higher perspective on elementary mathematics, Ball and Bass emphasized their Horizon Content Knowledge (HCK) as "a kind of elementary perspective on advanced knowledge" (2004, p. 10). It includes the understanding that gives teachers peripheral vision for the teaching context and their pupils' current level of understanding, which allows instructors to be aware of the effects of mathematical concept representations on future learning (Ball & Bass, 2009; Ball et al., 2009). An example of Sean numbers is given by Ball and Bass:

One day, a third grader named Sean proposed his idea that 6 can be an even and an odd number at the same time because it can be divided into three groups of two or two groups of three (6 = 2 + 2 + 2 or 6 = 3 + 3). The mathematics teacher in his classroom did not immediately correct him; instead, the teacher invited the class to comment. Sean's classmate, Mei, did not agree with him but helped him generalize his idea to odd multiples of two (i.e., in three groups of two, "three" is an odd number).

From the example, Ball and Bass recognize that the class has implicitly discussed three types of definitions of even and odd, i.e., Fair-share, Pair and Alternating. In Fair-share, a number is even if it can be made into two equal groups with none left over. In addition, Sean numbers also revealed a glimpse of mod 4 arithmetic (identifying numbers as congruent to 2 mod 4; e.g., 6). More significantly, these numbers were studied by the ancient Greeks: the squares mod 4 are 0 and 1, so the differences between two squares mod 4 are 0, 1, and -1 (or 3), but not 2. Ball and Bass tried to use this example to argue that students' ideas may sound wrong on the surface, but when using horizon knowledge, a teacher can nevertheless hear something insightful.

However, Ball and colleagues' work on HCK was limited by the vagueness of its definitions and lack of more concrete examples. In contrast, Zazkis and Mamolo (2011) provided specific examples and extended the concept of horizon knowledge to teachers' horizons (from advanced to elementary) in addition to the learners' horizons (from elementary to advanced) identified by Ball and colleagues, with inner horizons based on the mathematical object and outer horizons external to the mathematical object. HCK was defined in terms of using undergraduate mathematics or advanced mathematical knowledge in the current teaching. In the recent article, HCK is further defined as enabling teachers "to make judgments about the importance of particular ideas or questions, and to treat the discipline with integrity, all resources for balancing the fundamental task of connecting learners to a vast and highly developed field" (Jakobsen, Thames, Ribeiro, & Delaney, 2012, p. 4642)

An example (Figure 1) of HCK was shown in the Zazkis and Mamolo's (2011) article: A teacher asked her 3rd Grade students to count how many triangles were in Figure 1. The students were provided two answers, 32 and 27. The teacher though,

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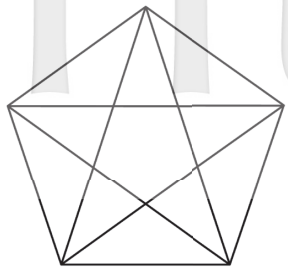

Figure 1. How many triangles

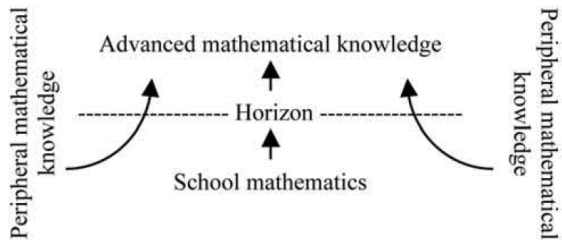

Figure 2. Peripheral mathematical knowledge

Source: "Peripheral mathematical knowledge," by C. Foster, 2011, *For the Learning of Mathematics*, *31*(3), pp. 24-26.

"she had not yet counted herself, immediately knew that both answers were incorrect" (2011, p. 8). It is because "she recognized rotation symmetry of order 5 in the figure and, as such, she knew that the number of triangles should be a multiple of 5" (p. 10). Having knowledge about "rotation symmetry of order 5" (advanced mathematical knowledge) is the HCK in this case.

Foster, which contributed the idea of "peripheral mathematical knowledge" to the HCK model (Figure 2), viewed peripheral mathematical knowledge as "coming in from the sides, cushioning and supporting the learner's mathematical trajectory upwards" (2011, p. 24).

Peripheral mathematical knowledge is a sense or a feeling about mathematics knowledge. For example, when someone offers a conjecture and the teacher does not know whether it is true, peripheral mathematical knowledge will give the teacher a sense by which to judge the conjecture. Peripheral mathematical knowledge is also some mathematical fact that students may or may not be expected to learn but it is useful for teachers in teaching. It may include, for example, such fact as "a cube has 9 planes of symmetry" or "$\frac{19}{95}$, $\frac{16}{64}$, $\frac{49}{98}$, $\frac{26}{65}$ and their reciprocals are the only 2-digit over 2-digit fractions that give equivalent fractions by cancelling off identical digits." Knowledge like these helps teachers know how to guide students. Peripheral mathematical knowledge is not only about memorizing these facts but also about applying them in teaching. For example, when a teacher knows the answer is 35 (mathematical fact) in Figure 1, he or she would make teaching decisions more fluently (e.g., directing students toward systematic counting and cataloguing of different shapes). It is because the teacher is familiar with the teaching situation concerning the problem. In addition, peripheral mathematics knowledge varies from teacher to teacher, depending on the background of the students who teachers work with.

Although Zazkis and Mamolo's (2011) and Foster's (2011) work deepened the understanding of HCK, Figueiras et al. (2011) criticized the idea that advanced mathematical knowledge was directly applied to teaching situations because "mathematical problems behave very differently in a pure mathematical setting than in an educational context" (Figueiras et al., p. 26). For example, Figueiras et al. noted that teachers know two different approaches for defining a fraction from part-whole and division, although these two meanings are not so trivial for students. More importantly, Figueiras et al. claimed that this aspect of ad-

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vanced mathematical knowledge might also lead to students' misunderstanding of a mathematical concept.

In our opinion, HCK includes not only applying advance mathematical knowledge but also using the very general mathematics knowledge on the current teaching topics. This general mathematics knowledge is "fundamental mathematical knowledge."

Fundamental Mathematical Knowledge

Zazkis and Mamolo (2011) defined HCK as the use of advanced mathematical knowledge on the current teaching object, and Foster (2011) added the use of mathematics facts (or senses) to the HCK model. However, their ideas did not include Ball and Bass's (2009) description of "a kind of elementary perspective on advanced knowledge." In fact, Zazkis and Mamolo attempted to stress the teachers' horizon (from advanced to elementary) in the HCK model. From the perspective of Ball and colleagues' point of view, HCK may emphasize valuing the current teaching topic and considering how this topic will relate to future mathematics learning. In this study, we try to integrate both ideas, but as we mentioned earlier, the concept of HCK given by Ball and colleagues remains vague. In addition, their conception of HCK is much related to a sense of the importance of students' arguments during the classroom discussion. However, we believe that there exists a kid of knowledge which is basic and truly reflecting the Ma's (1999) idea of "longitudinal coherence." We therefore propose the concept of "fundamental mathematical knowledge" (FMK) to more precisely capture the concept of HCK from their definition (or at least, to conceptualize an idea compatible with their proposed version of HCK).

FMK is defined as the ability to recognize and emphasize essential mathematics principles or ideas when teaching problems or concepts. It is a type of profound and penetrative pedagogical thinking that inspires awareness of basic, primary, core knowledge beyond simply teaching problems or concepts by rote memorization. It is the very general mathematics principles or ideas which the mathematics knowledge or problems could develop from. This type of knowledge may help students connect to their intuitive thinking due to its emphasis on the basic elements of mathematics. It may also enable students to generate broader insights beyond rote learning because these basic yet essential ideas extend beyond the procedural knowledge necessary to solve a particular problem or to understand a specific concept.

More importantly, FMK provides a consistent view on different grade levels of problems. We may use the "hard core" (Lakatos, 1978) and the "theory of everything" (proposed by Einstein) as analogues of FMK. FMK, just like the hard core, is important and never changes, while it also can, as the theory of everything does, explain all the related problems by only using one key idea.

Evidence for fundamental mathematics knowledge

Our idea of FMK is influenced by Profound Understanding of Fundamental Mathematics ([PUFM]; Ma, 1999) -- in particular, its "basic ideas" (simple but powerful) and "longitudinal coherence." PUFM is built on the research which

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focuses on the depth of teachers' subject matter knowledge (Ball, 1989; Grossman, Wilson, & Shulman, 1989; Marks, 1987; Steinberg, Marks, & Haymore, 1985; Wilson, 1988). The PUFM follow-up studies have already shown that teachers require deeper and broader knowledge to effectively teach mathematics (Beckmann, 2002; Chick, Baker, Pham, & Cheng, 2006; Hossain, Mendick, & Adler, 2013; Schoenfeld & Kilpatrick, 2008). For example, Vale, McAndrew and Krishnan (2011) indicated that their research program could help teachers develop deeper and broader mathematics knowledge, and with the specialized mathematics knowledge, teachers could then develop their practice to support students' present and future mathematics learning. Since PUFM is theoretical support for FMK, we then further discuss their relationship in the following section.

Profound Understanding of Fundamental Mathematics and Fundamental Mathematical Knowledge

PUFM, proposed by Ma (1999), regards mathematics teachers' conceptual grasp of elementary mathematics. It can be defined as follows:

> [PUFM] is more than a sound conceptual understanding of elementary mathematics—it is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instill those basic attitudes in students (1999, p. 124).

There are two key ideas given by Ma's book: (1) knowledge packages and (2) the four characteristics of teachers' PUFM. "Knowledge packages" refer to the composition of "a refined sense of the organization and development of a set of related ideas" (Ball, Lubienski, & Mewborn, 2001, p. 449). For example, Figure 3 is a model of the knowledge package for subtraction. The knowledge pieces in the package are not isolated and have different levels. The concept and procedure of subtraction with regrouping develops step-by-step from the base of "addition and subtraction within 10" to the top of "subtraction with regrouping of large numbers." The topic of "addition and subtraction within 20" is the key idea in this sequence (shape with the thick border). Furthermore, other necessary topics (basic principles) link directly or indirectly to the main sequence. Within them, "composing and decomposing a higher value unit" is another key piece in the package because it is the core concept underlying the subtraction algorithm.

Ma also described the four characteristics of PUFM: (1) connectedness, (2) multiple perspectives, (3) basic ideas and (4) longitudinal coherence. Connectedness means making connections among mathematical concepts and procedures. Connectedness helps students to learn a unified body of knowledge. "Multiple perspectives" refers to appreciating different facets of an idea and various approaches to a solution. Basic ideas are the awareness of the simple but powerful basic concepts and principles of mathematics. Longitudinal coherence means that teachers are not limited to the knowledge they are teaching in a certain grade. Instead, "they also know what students are going to learn later and take opportunities to lay the proper foundation for it" (Ma, 1999, p. 104).

In addition, Ma used a "proficient taxi driver"

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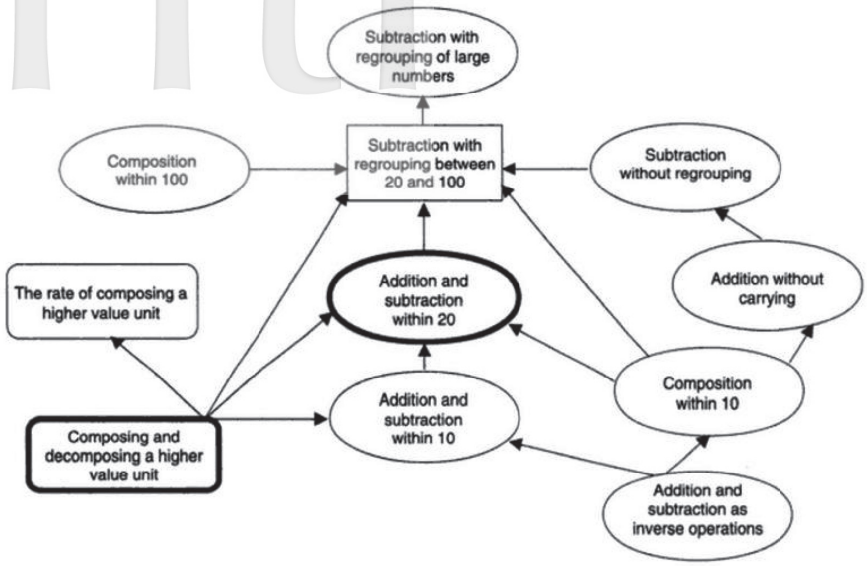

Figure 3. The knowledge package for subtraction

Source: *Knowing and learning mathematics for teaching: proceedings of a workshop* (p. 15), by National Research Council, 2001, Washington, DC: National Academy Press.

as a metaphor for a teacher equipped with PUFM: A proficient taxi driver knows all the roads in the town quite well. He or she can flexibly take different routes when needed. Moreover, a passenger does not even have to give complete information about the address; the taxi driver can still find the destination. A PUFM teacher operates in the classroom as the taxi driver operates in the town. A complex map of the school's mathematics is in the teacher's mind.

What is the difference between PUFM and FMK?

PUFM and FMK (a sub-category of HCK) share a common background. (1) They both build on Shulman's (1986) Pedagogy Content Knowledge (PCK). (2) Ball's concept of subject matter knowledge is more or less influenced by Ma's (Ball & Bass, 2000; Ball et al., 2001). (3) Roughly speaking, the essences of PUFM and FMK are the same. The two PUFM properties of "basic ideas" and "longitudinal coherence" are also stressed by FMK.

Based on the above similarities, we believe that if we use PUFM as a framework to analyze an FMK teacher, we will find the teacher also have PUFM. It is because both PUFM and FMK teachers seem to have a great sense of current and related teaching topic(s). However, the two concepts still have some distinct features:

First, PUFM is broader than subject matter knowledge, whereas FMK (as a sub-category of HCK) is located in the category of pure subject matter knowledge in the MKT model. Ball et al. comment on PUFM shows this difference: "Ma's notion of knowledge packages represents a particularly generative form of and structure for pedagogical content knowledge" (2001, p. 449). Therefore, how to "communicate with students" (Howe, 1999, p. 885) is also included in PUFM.

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The second difference may derive from their research method. PUFM is based on what teachers said in the interviews; however, MKT builds on practice-based observations, which is developed from carefully examining the teaching practice. Therefore, PUFM shows what the structure of teachers' mathematics knowledge may be, but it is unclear how it actually affects teachers' practice (Ball et al., 2001). In contrast, MKT is the knowledge needed in teaching, which means a greater focus on teachers' decision making and behaviors in the real practice (Ball et al., 2008).

Third, FMK is more compact than PUFM. PUFM incorporates multiple dimensions, such as deep, broad and thorough, while FMK merely focuses on bringing out the consistency behind problems at different difficulty levels.

Fourth, the "longitudinal coherence" in PUFM is more emphasized on the elementary curriculum but FMK is more for the advanced mathematics knowledge.

Fifth, the "basic ideas" in PUFM is focused on developing other mathematical concepts based on the "basic idea." On the other hand, the "basic ideas" in FMK means that when using the same "basic ideas," all the related problems across different difficulty levels are isomorphic (essentially equivalent). That is, PUFM emphasizes "developing," while FMK is more centered on "consistency."

Method

This study is a qualitative research. We adapted a video-based interview method to capture an experienced teacher's FMK (Šunderlík & Ceretková, 2012).

FMK is conceptualized using the practice-based approach. The best way to find teachers' FMK is to conduct a longitudinal research by carefully analyzing lots of teaching videos, similar to what Ball and colleagues have done. However, as an initial stage in exploring the concept of FMK, we should seek a more efficient method. Video has been considered as a popular practice-based medium for a long time (Seago, 2003), and Kersting (2008) has shown that using videos is a good way to explore teachers' knowledge, particularly MKT.

Therefore, to examine our conceptualization of FMK, we adopted a method using video clips of classroom instruction as a prompt to elicit the participant (John, with pseudonym) to demonstrate his knowledge of FMK. We called this process video-based interviews. In this approach, the participant was asked to analyze the selected classroom teaching video clips and discuss them with the rest of the research group members (interviewers). The research group asked questions to clarify John's opinions and to criticize his ideas during the interviews to outline John's FMK and also to triangulate his arguments. All of the discussions were transcribed and then analyzed. The results are provided as examples in the later section.

The participant

John, who is a retired mathematics teacher, taught for 39 years in a top-ranking high school in Taiwan. Before he formally participated in our study, he was invited to join our group meeting as a consultant. The purpose of those meetings was to analyze teachers' teaching in the other domains of MKT (not FMK). When he commented on those teachers' teaching, he seemed to demonstrate a broader view on them, which made us curi-

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ous whether he is a teacher with FMK. From the research point of view, experienced teachers are more likely to have better MKT (Herbst & Kosko, 2012). Therefore, we invited John as our participant because we conjectured that an experienced teacher, such as John, would most likely have more opportunities to develop FMK.

Research group

During the video-based interviews, John was interviewed by the research group. The research group was comprised of a tertiary mathematics teacher educator, a postdoctoral fellow, two Ph.D. students and two graduate students. In this group, one of the Ph.D. students and one of the graduate students were current high school mathematics teachers with roughly eight years of teaching experience. Members of the research group all received rigorous training in mathematics and mathematics teaching and, in particular, were familiar the theory about MKT.

Video clips

There were a total of 11 video clips. Each video is approximately 20 minutes. The content of these videos is the teaching practice of three experienced, in-service high school mathematics teachers. They served as examples of how teachers generally teach high school mathematical topics. Of the example teachers, Chang earned a Ph.D. in mathematics, and the other two teachers (Weng and Hu) earned B.S. degrees in mathematics. They were all very experienced and had high-level backgrounds in mathematics and mathematics teaching methods (as evidenced by the high grades they received when graduating from the normal university). In addition, they all taught at top-ranked public high schools in the local district.

We selected "vectors and 3-D geometry" and "mathematical induction" as our focus because, in the analyses of curriculum materials and previous research reports (e.g., Kösa & Karakuş, 2010), these areas comprised a significant portion of the high school curriculum (e.g., vectors and 3-D geometry take nearly an entire semester in Taiwan), and difficulties existed in the teaching and learning of these topics.

Data analysis

Table 1 shows the coding schema we used in this study. It is developed based on our conception of HCK, including the codes of advanced knowledge, core knowledge, connections and so on. These codes are indicators of teachers' HCK.

Table 2 shows how an excerpt was coded in this study. The excerpt in Table 2 was Mr. John's critique on Teacher Chang's teaching of mathematical induction. The coding results helped us to know when the HCK appeared and, in particular, how many different FMKs were mentioned.

Examples of Fundamental Mathematical Knowledge

Example 1

When teaching Heron's shortest distance problem (Figure 4), teachers should focus on the basic principle, "The shortest distance between two points is a straight line."

The typical way to teach this problem is to find the point A' that is symmetric to A with respect to line L and to notice that the line

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Table 1. Coding schema

Category	Meaning
Advanced Knowledge (AK)	The teacher refers to advanced mathematics knowledge about teaching the topic.
Core Knowledge (CK)	The teacher refers to the basic or core mathematics knowledge about teaching the topic.
Connection (CN)	The teacher refers to how the current topic is related to further or earlier topics.
Critique (CT)	The teacher gives comments or critiques to the teaching in the video.

Table 2. An example of how a transcript was coded

Excerpt	Coding
After having introduced a problem proved by the mathematical induction, Teacher Chang told students: when you get n = 1 is true, then you can get n = 2 is true; when you get n = 2 is true, then you can get n = 3 is also true. [Teacher Chang continually provided examples until n = 7].	
Mr. John: Pause [the video] here. I think there is a flaw here. You cannot use specific examples to produce another. It should be a generic rule for any terms.	CT
Interviewer 1: What do you mean "generic" here? Do you mean using n = 1 to generate n = 2 and n = 2 to generate n = 3 is not correct?	
Mr. John: No, as I said earlier there are two critical principles related to mathematical induction. One is the beginning and the other is the continuing. Regarding the continuing here, you cannot explain by a specific n, like n = 1 or n = 2, whatever. It should be regularity for any $n = k$ and $n = k + 1$. When the regularity is held, then you can say it is true for n = 1 and n = 2 or other similar patterns.	CK
Interview 2: Why do you so stress on clearly explaining [the continuing]?	
Mr. John: If you use a specific term to show your "continuing" how do you make sure they are always true for the remaining terms? Especially, they are infinite. And if you do not precisely introduce the two basic ideas for students, I don't think they will fully understand.	CT CK
Interview 1: Since you mention students, could you say more about how to teach it?	
Mr. John: I think you could provide more concrete examples... [the examples are omitted here] I remember a student asked me "can we use mathematical induction inversely," which means we use $n = k$ to induce $n = k$ - 1 and so on.	AK
Interviewer 3: Can we?	
Mr. John: Of course, we can, but the question is why you need to do that? Usually, we don't need to prove anything backward but in some cases, like the proof of inequality of arithmetic and geometric means. Once you have proved n = 2, 4, 8... are true, you can prove n = 3 backward (see the example 3 for detail). This is what I said: if you hold the two basic principles (beginning and continuing), this solution is easy to comprehend. You can easily connect it to a backward case of the mathematical induction method. However, if you don't build the mathematical induction method firmly in your mind, the problem is novel to you and you can solve it because you memorize the solving method.	AK CN

joining B and A' intersects L at P, which is the point that provides the solution. To explain the meaning of "finding the point that is symmetric to A with respect to line L", teachers usually use a formal proof with the triangle inequality (Figure 4).

However, this type of teaching may result in frustration among students when solving advanced problems (e.g., Figure 5) because the teacher does not help students to recognize the basic principle (or core idea), "The shortest distance between two points is a straight line." What is the difference when the teacher emphasizes the basic principle in teaching these problems? The teacher might initially reduce the shortest distance problem to involve two points on "different sides" of a given line - the case in which two points and a line lie in the same plane. This might enable students to

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reduce the consideration to joining two points with a straight line and to understand that A is reflected as A' to preserve its distance. The teacher could help the students to appreciate the relationship between the extended three-dimensional problem and the original problem because the symmetry in the original problem also involves a spatial rotation of 180° around L (Figure 6). When taking FMK into account, the original and extended problems are essentially equivalent.

Example 2

When teaching the smallest volume problem (Figure 7) in Taiwanese high school, the fundamental mathematical principle is "P is the centroid."

From our study (Chin, 2011), many high school teachers ignore "P is the centroid" in teaching this problem. A teacher even said that teaching this problem is meaningless because he felt that the problem only allows students to apply the formula. However, in fact, the "center property" for minimizing volumes is crucial when teaching the problem because it will hold from 2 to n dimensions, even for axes that are not perpendicular (e.g., an oblique coordinate system). For example, the original smallest volume problem could be reduced to a two-dimensional case: "Of all lines through the point P, find the line that cuts off the triangle of smallest area in the first quadrant." The smallest area occurs only when P is the midpoint of $\overline{AB}$ (Figure 8). In addition, the center property can be extended "through a point inside a circle or a parabola".

Example 3

When teaching the "mathematical induction" in Taiwanese high school, rather than presenting it as a set list of steps for students, teachers should focus on the two basic (core) principles of "base" and "induction." "Base" means that there is a starting point for a statement (usually $n = 1$), and "induction" means that if the statement holds for

Find a point on a line L, the sum of whose distances to two given points A and B on the same side of the line is the minimum.

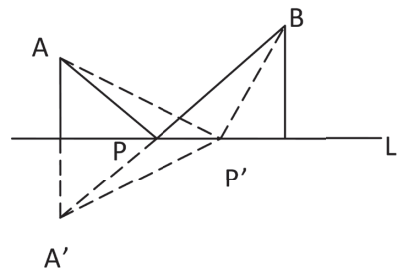

Figure 4. Heron's shortest distance problem

Find a point on a line L "in space," the sum of whose distances to two given points, A and B, is the minimum.

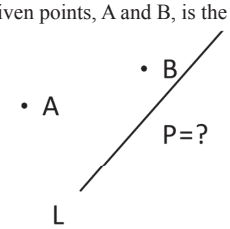

Figure 5. The extended problem ($\overleftrightarrow{AB}$ and L do not intersect with each other)

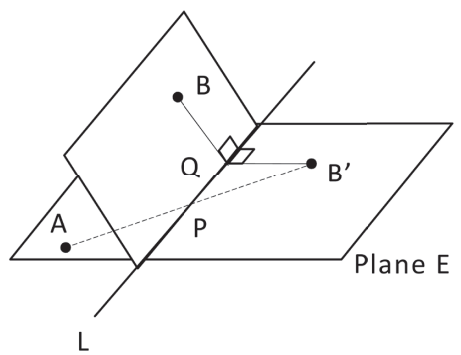

Figure 6. The solution to the extended problem

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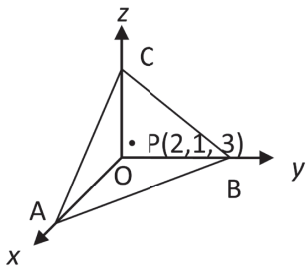
Problem: Plane E passes through point P (2, 1, 3) and cuts off the smallest volume in the first octant. (a) Find the smallest volume of the tetrahedron cut by the plane. (b) Find the equation of the plane.



Teaching process:

(a) Let plane E be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$*; because (2, 1, 3) lies on plane E, we have* $\frac{2}{a}+\frac{1}{b}+\frac{3}{c}=1$.

(b) The inequality of arithmetic and geometric means gives $\frac{\frac{2}{a}+\frac{1}{b}+\frac{3}{c}}{3}\geq\sqrt[3]{\frac{2}{a}\times\frac{1}{b}\times\frac{3}{c}}$*;*

therefore, V(a, b, c)= $\frac{1}{6}abc\geq\left(\frac{3}{\frac{2}{a}+\frac{1}{b}+\frac{3}{c}}\right)^3\geq 27.$

(c) Now, we notice that (a, b, c)=(6, 3, 9) gives equality in this inequality; thus, the equation of plane E is $\frac{x}{6}+\frac{y}{3}+\frac{z}{9}=1.$

(d) Next, let us replace 2, 1 and 3 with x_0*,* y_0 *and* z_0*; by the same procedure, we get (a, b, c)=(*$3x_0$*,* $3y_0$*,* $3z_0$*), and the plane is* $\frac{x}{3x_0}+\frac{y}{3y_0}+\frac{z}{3z_0}=1.$

(e) P is the centroid of ΔABC.

Figure 7. The smallest volume problem and its solution

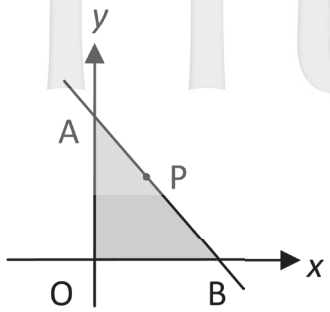


Figure 8. The center property in two dimensions (P is the midpoint of $\overline{AB}$)

some $n = k$, $k \in N$, then it also holds for $n = k + 1$.

It is typical for teachers to focus on the rote steps of mathematical induction. However, teachers who consistently stress the rationale (base and induction) may help students to comprehend that the "base" is not always $n = 1$ or only one beginner (e.g., the proof of the Fibonacci sequence has two beginners), and "induction" can work in reverse (reverse induction), such as the proof of inequality of arithmetic and geometric means:

After the case of "$n = 2, 4,...2^k..., k \in N$" is initially proved, then the remaining case of "$n \neq 2^k$ and $n < 2^k$" can be proved backward. For example, when $n = 4$, $\frac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt[4]{a_1 a_2 a_3 a_4}$ is proved; then, we prove backward to supplement the case of $n = 3$:

$$\text{Let } d = \frac{a_1 + a_2 + a_3}{3} \Rightarrow \frac{a_1 + a_2 + a_3 + d}{4} \geq \sqrt[4]{a_1 a_2 a_3 d}$$

$$\Rightarrow \frac{3d + d}{4} \geq \sqrt[4]{a_1 a_2 a_3 d}$$

$$\Rightarrow d \geq \sqrt[3]{a_1 b_2 c_3}$$

From above examples, we could find that FMK is a type of knowledge that involves what teachers value and specific ways that teachers teach mathematics. It is broader than a heuristic because it deals with more issues regarding complex thought about teaching. For instance, in Example 1, when teachers possess "the shortest distance between two points is a straight line", they could teach in another way that would help students to better connect to their intuitive thinking and to later (advanced) mathematics learning. That is, helping students to see two points on "different sides" of a given line connects more closely to their intuitive thinking and helps students to realize that symmetry in the original problem involves a spatial rotation of 180° around L, thereby advancing their learning from 2-D to 3-D.

In Example 2, although teachers could solve the problem without invoking this "center property," they would be less likely to provide students with a mathematically useful version that identified connections within the "larger mathematical landscape" (Ball & Bass, 2009, p. 1).

Rethinking HCK

A revised model of HCK that links advanced, peripheral and fundamental mathematical knowledge is provided (Figure 9).

When HCK is defined as the understanding of a "broader set" of mathematical ideas connected to a particular mathematical concept (Ball et al., 2009), three types of broader sets should be involved. The first type is an advanced perspective on elementary knowledge (advanced mathematical knowledge). Complementary to the advanced perspective, there is also an elementary perspective on advanced knowledge (FMK). Finally, knowledge of some mathematics facts beyond the current teaching (Peripheral Mathematical Knowledge [PMK]) is used to support both advanced and fun-

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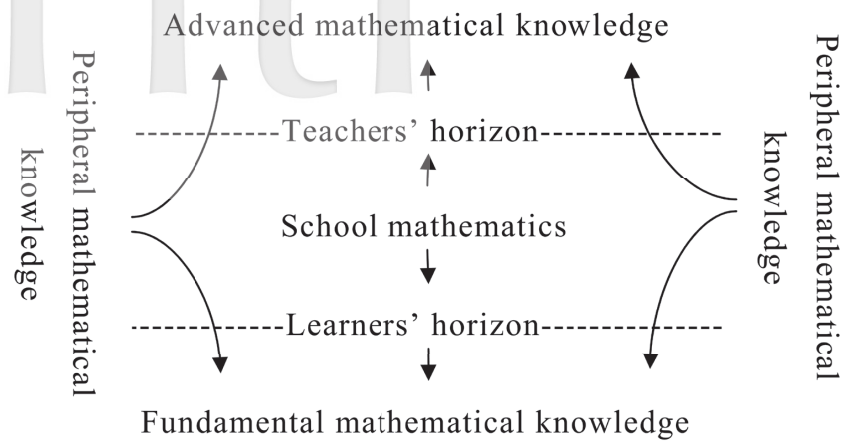

Figure 9. A revised model of horizon content knowledge

damental mathematical knowledge. "Advanced" and "fundamental" are along the difficulty levels of topics being taught, while "peripheral" is not. "Peripheral" is the teachers' sense of the topics being taught based on the teachers' experience (familiarity with the teaching process).

This model has been revised in Figure 9 based on Foster (2011). The school mathematics now moves to the center. In the upward direction, the level of mathematical sophistication is increased, and there is a fruitful interaction between the teachers' Advanced Mathematical Knowledge (AMK) and the school mathematics. This axis represents applying advanced mathematics knowledge to the current teaching topics, which is called the teachers' horizon by Zazkis and Mamolo (2011). We use this visual representation to try to reflect Zazkis and Mamolo's definition that "The horizon being 'farther away' for the teacher enables him or her to see more features and attributes of an object, and to gain a more in-depth appreciation for what exists in the outer world" (p. 10).

In the opposite direction, it is the learners' horizon that focuses on "a kind of elementary perspective on advanced knowledge that equips teachers with a broader and also more particular vision and orientation for their work" (Ball & Bass, 2009, p. 10). The progression downward indicates seeking the basic but critical ideas in the learners' school mathematics. In our definition, it is FMK. With the simple but central idea in mind (the awareness of the consistency of a series of problems), all the related problems at different grade levels (even at the college level) can be observed as isomorphic or essentially equivalent. Although Figure 9 does not show a clear connection between FMK and AMK, we assert there is a complex interaction between the two. For example, a teacher knows that a basic idea is critical to a problem (FMK) because the teacher realizes this specific idea is consistently useful in advanced mathematics (AMK).

Foster (2011) defined PMK as knowing some mathematical fact or a type of mathematical sense to help boost the learners' mathematical trajectory upward. PMK is another type of broader view of HCK, but the broader view does not come directly from increasing the difficulty of school mathematics (AMK) or delving down to its foundation

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(FMK). So what is the difference between PMK and AMK? How do these two notions work together? Our understanding is that PMK supports AMK in applying the advanced knowledge. In the case of the "how many triangles?" problem above (Figure 1), AMK tells the teacher the number of triangles should be a multiple of 5 based on knowledge of rotational symmetry and its order so that the teacher can guide the students to count different types of triangles systematically and realize there are 5 of each the same type of triangle. PMK means the teacher knows the correct answer is 35 (because the teacher is familiar with the problem). This knowledge will help the teacher to judge student answers immediately (though the teacher can also use "a multiple of 5" to judge incorrect answers). In addition, PMK helps the teacher develop a sense that not all regular polygons will share the same rule. For example, a triangle has three sides but just one triangle. A hexagon has six sides but includes a total of 110 triangles (not a multiple of six). PMK will not give teachers a formal proof but will provide an initial sense of whether a conjecture is true or not (Foster, 2011). It is an immediate reaction and most likely arises from the teacher's experience and other types of subject matter knowledge (e.g., AMK, FMK, Specialized Content Knowledge [SCK]; Ball et al., 2008). Foster further explains the source of his PMK in this problem: "My experience with counting things in symmetrical arrangements has led me to expect complexity, and the idea that five-fold symmetry implies that the number of triangles must be a multiple of 5 seemed too easy" (Foster, 2011, p. 25). Along this line, we believe that PMK can also work with FMK (see an example in the next paragraph). However, we would like to argue that each category in the HCK model may also work individually, which means that when AMK or FMK is active, it is not always associated with PMK. Either AMK or FMK can also work alone.

In addition, AMK, PMK and FMK are relatively new ideas in the research field. We try to build a framework of HCK by incorporating those three elements. In particular, we focus on the FMK in this article. Much work remains to be done to better understand their effect on teaching and to develop more sophisticated definitions of those three terms.

We take the shortest distance problem (Figure 4) as an example to illustrate the above revised model. When teaching the shortest distance problem (school mathematics), a HCK teacher knows what the extended problems or learning (advanced mathematics knowledge) could be and what the most important core knowledge (fundamental mathematics knowledge) for solving this series of problems is. Peripheral mathematical knowledge in this example could be knowing the procedure for solving the problem immediately (i.e., reflecting A across L leads to the solution). Equipped with these different types of knowledge in HCK, the teacher finally decides that he or she should focus on teaching the basic principle instead of only teaching how to solve the problem.

A Sample Item

Developing a measure of FMK is an important step to move our conceptualization into practice. It is not only meaningful for advancing our understanding of FMK, but the results from pilot-testing the FMK measure can also help assess whether our conceptualization is correct and adequate (Hill, Ball, & Schilling, 2008).

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From our definition of FMK, there have several key properties that should be included in the test items. (1) It arises from practice. (2) It is a type of depth and breadth of knowledge that goes beyond the current teaching topic. (3) It concerns the basic but critical ideas in teaching. (4) It relates to advanced mathematical knowledge. Therefore, in our item design, we begin with a teaching scenario (adapted from a real classroom teaching) in which a teacher addresses a problem without FMK. We ask the respondents to identify the most important basic idea in teaching this problem to examine if the respondents have the relevant knowledge. We also invite the respondents to critique the teaching in the scenario to elicit the teachers' FMK. In addition, we pose questions asking if the respondents know what might be the extension of the problem, which is related to their advanced mathematics knowledge. Finally, our item adopts an open-ended design based on Hill's et al. (2008) suggestion that researchers can use open-ended items to avoid deviation from the multiple-choice format.

Figure 10 presents a sample item that illustrates how teachers' HCK might be captured. The item is based on analyses of numerous classroom observations. Further information on the item development process is found in Lin and Chin (2014). The reliability and validity of open-ended questions depend on whether the respondents interpret and respond to the question in the same way as we intended. According to Lincoln and Guba (1985), we could examine the trustworthiness of item construction by credibility, transferability, dependability and confirmability. Therefore, group discussion with multiple background experts, pilot-testing items and following-up interviews were used in this study to achieve the above criteria.

The item design consists of three components:

Problem: Find a point on a line L, the sum of whose distances to two given points A and B on the same side of the line is the minimum.

Mr. Weng teaches his students to solve the problem by reflecting A across L to obtain point A': the point of the intersection of L with $\overline{A'B}$ *is the answer.*

Q1. What is the most important mathematical principle needed to solve this problem?

Q2. Compared to Mr. Weng, your teaching of this problem is □ Roughly the same □Not roughly the same but not completely different □ Completely different.
Explain:

Q3. What approaches might you take to enable students to appreciate that reflecting point A across L leads to the solution?

Q4. What might be some mathematical extensions of this problem (e.g., what might be an extended problem, and how would an extended problem influence your teaching of the initial problem)?

Figure 10. The shortest distance item

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(1) a geometric problem with a solution that can be generalized, (2) a usual teaching approach and (3) a series of probe questions to assess teachers' HCK. In addition, the probes examine three types of information: Q1. teachers' understanding of the FMK to be used to solve the problem, Q2. teachers' attitudes (thinking) toward a particular teaching approach and Q3 and Q4. teachers' practices for the extended problem.

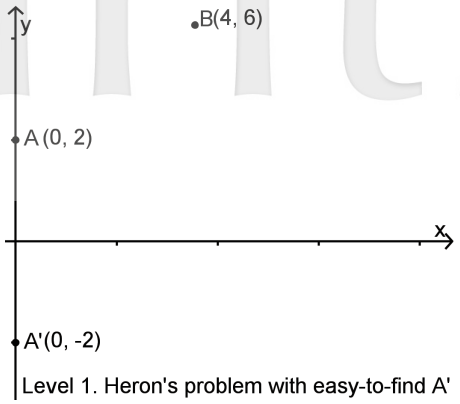
In this item, we use prompts to solicit teachers' comments or reflections on a typical teaching approach to the shortest distance problem. We anticipate that a HCK teacher could indicate the "straight line principle" in Q1; employ another teaching approach which focuses more on the "straight line principle" in Q2; explain reflecting point A not only by the triangle inequality but also by something related to the "straight line principle" in Q3; and indicate the possible extended problems and how it relates to the teaching in Q4.

Discussion and Conclusion

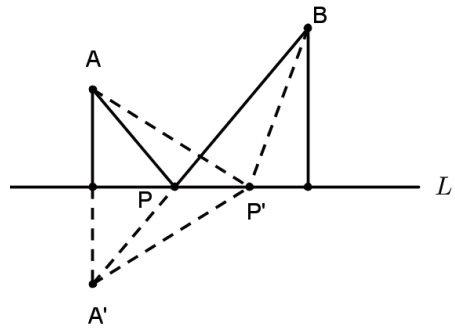
Based on prior research in the field of HCK (Ball & Bass, 2009; Foster, 2011; Zazkis & Mamolo, 2011), this article expanded the conceptualization of HCK and complemented earlier models by incorporating the dimension of FMK. As a dimension of HCK, FMK is not knowledge that the students would be expected to learn either at present or in the future (Ball & Bass, 2009), but the knowledge that teachers might use to enable students to deal with mathematical problem more generally and achieve a more comprehensive understanding that is not restricted to the procedures used to solve a particular problem.

Here are more discussions about the "shortest distance" principle as "a kind of elementary perspective on advanced knowledge" (fundamental mathematics knowledge). In the shortest distance problem, when teaching is based on "the shortest distance between two points is a straight line," students can easily understand that the series of problems in Figure 11 are essentially equivalent (we discuss only level 2 and level 4 problems in the above section). However, these problems could be located in different teaching areas and grade levels. Level 1 and level 2 problems, which are suitable for junior high school students (in Taiwan), are located in the teaching area of lines. Level 3 and 4 problems, which are suitable for high school students, are located in the teaching areas of lines and planes in space (geometry). For junior high school students, high school mathematics can be a kind of "advanced knowledge," where "advanced" refers to different grade levels rather than to college mathematics knowledge. Nevertheless, we believe that if students can understand the essence of these geometry problems, this may help their future learning in college geometry.

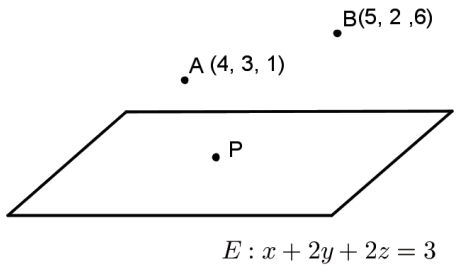
Is recognizing different levels of the shortest distance problem a type of Knowledge of Content and Students (KCS)? We tend to think it is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum," which is HCK (Ball et al., 2008, p. 403). Therefore, junior high school teachers need to know how the topic being taught is related to what students will learn in the high school in the above case. Having this horizon knowledge will help teachers to decide how to teach the shortest distance problem. KCS is more related to understanding students' common conceptions and misconceptions about a particular mathematical concept (Ball et al., 2008). For example, what are students



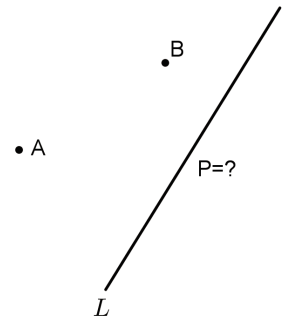
Level 1. Heron's problem with easy-to-find A'



Level 2. Heron's shortest distance problem



Level 3. Heron's problem via a plane



Level 4. Extended Heron's problem

Figure 11. Different levels of short distance problem

likely to think about the shortest distance problem? What common errors are the students most likely to make in this problem? However, Ball et al. also acknowledged that "It is not always easy to discern where one of our categories divides from the next" (Ball et al., p. 403). Therefore, it may, at times, not be easy to assert that a teacher's knowledge is purely HCK.

Because FMK may affect teaching problems, it may seem to be a heuristic for problem solving. However, we argue that FMK is broader than a heuristic. First, FMK is more closely related to teaching. It represents the knowledge that teachers hold. Based on this knowledge, teachers may teach a problem in a specific way or focus on critical principles or ideas that provide insight for students in later (or advanced) mathematics learning. Second, FMK can be used not only in teaching problem solving but also in teaching mathematics concepts, such as mathematical induction (see above section).

FMK focuses more on to how to teach problems or concepts than on problem-solving strategies. It involves complex pedagogical thinking and teaching decisions, such as how to design a teaching process that increases students' awareness of basic principles of mathematics ideas. There are multiple ways to achieve this intention to emphasize basic principles. For example, teachers could use a formal proof for the triangle inequality (the

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most typical way) to teach the shortest distance problem (see above section), but they could also emphasize "the straight line principle" in some way (however, most teachers who use the formal proof do not emphasize the principle). In addition, teachers could use another approach to teach the "straight line" principle: If we assume that the line lies on a mirror plane, the problem becomes one of determining the path from A to B through a mirror that has a minimum optical length. Based on the equal angle law of reflection, it will make sense for students to find the mirror image of A. In addition, this law implies that light always takes the shortest path. Different approaches could have the same effect in emphasizing basic principles.

As Mr. John pointed out in this study, a teacher should not only teach mathematics by providing strategies but also be able to provide the primitive idea behind those strategies that can help students develop a broader view and conceptual understanding on the current topic. We have provided three concrete examples to show Mr. John's opinions. Through these examples, we hope to provide another point of view on teachers' HCK.

If we want teachers to develop HCK, then developing HCK assessment items is important. In this article, we show a HCK assessment sample item to measure the extent to which teachers applied this kind of knowledge. We believe that this article is a precursor to the design and implementation of large-scale studies. In addition, developing HCK assessment items that measure classroom practice would further illuminate and refine the conceptual framework of HCK. We also recommend that future studies examine how this knowledge is employed in the classroom and explore how to facilitate its use in teachers who do not currently possess it.

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擴展教師眼界數學知識——基礎數學知識與一個評量範例

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摘要

「眼界數學知識」(horizon content knowledge)意謂教師對目前所教授的數學內容，有更廣闊與深層的瞭解。本研究旨在提出「基礎數學知識」(fundamental mathematical knowledge)，藉以精緻「眼界數學知識」之架構；並呈現一個評量教師基礎數學知識之範例問題。基礎數學知識定義為：對所教授之數學內容或數學問題，具有深層、透徹且融會貫通的理解。當教師具有基礎數學知識時，能夠察覺 (awareness)教學主題中，核心且最基本的數學知識 (洞察問題背後的一致性)。掌握這個核心且基本的數學知識，可幫助學生融會貫通整個數學問題，有助於學生解決更高難度的相關數學問題，並幫助未來更高階數學的學習。基礎數學知識是從基礎的角度來看待高階數學 (elementary on advanced)。此外，發展基礎數學知識的評量，不但可幫助教師提升基礎數學知識，並能透過實徵的研究資料，健全基礎數學知識的定義。

關鍵詞：評量、基礎數學知識、眼界數學知識、教數學所需的知識、教師知識

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